

Gravitational Duality Transformations on (A)dS₄

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ABSTRACT: We discuss the implementation of electric-magnetic duality transformations in four-dimensional gravity linearized around Minkowski or (A)dS₄ backgrounds. In the presence of a cosmological constant duality generically modifies the Hamiltonian, nevertheless the bulk dynamics is unchanged. We pay particular attention to the boundary terms generated by the duality transformations and discuss their implications for holography.

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1. Introduction and Summary

Duality has played an important role in our understanding of Yang-Mill theories and it is believed that it will play an important role also in gravity and in higher-spin gauge theories. Indeed, although it is less clear what could be the implications of duality for theories whose quantum versions are still unknown, gravity and higher-spin gauge theories¹ are intimately connected to a quantum string theory where certainly duality plays a crucial role.

The recent advent of holography raises some intriguing questions for duality. For example one may wonder what is the holographic image of a duality invariant spectrum, a duality transformation or a possible quantization condition that usually duality implies for charges. Some of these issues were raised by Witten in [2] where it was argued that the standard electric-magnetic duality of a $U(1)$ gauge theory on AdS_4 is responsible for a “natural” $SL(2, \mathbb{Z})$ action on current two-point functions in three-dimensional CFTs.² Shortly afterwards it was shown in [3] that such an $SL(2, \mathbb{Z})$ action is intimately related to certain “double-trace” deformations in the boundary,

¹For reviews of higher-spin theories see e.g. [1].

²See [4] and [5] for more recent works.

assuming suitable large- N limits and existence of non-trivial fixed points. The latter assumptions are strengthened by the fact that there exist models (e.g., see [6] and references therein) which exhibit the required behavior. In particular, it was shown in [3] that certain "double-trace" deformations induce an $SL(2, \mathbb{Z})$ action on two-point functions of higher-spin (i.e. spin $s \geq 2$) currents. This has led to the *Duality Conjecture* of [3]: *linearized higher-spin theories on AdS_4 spaces possess a generalization of electric-magnetic duality whose holographic image is the natural $SL(2, \mathbb{Z})$ action on boundary two-point functions.*

Surprisingly, even the duality for linearized spin-2 gauge fields (linearized gravity) was not widely known by the time of this conjecture.³ Second order linearized gravitational duality was discussed among other in [8, 9, 10, 11, 12]. More recently, the duality properties of linearized gravity around flat space were studied in [13] and were further discussed in [14]. The duality of linearized gravity around dS_4 was later studied in [15].

In this note we present our calculations regarding the duality properties of gravity in the presence of a cosmological constant. Having in mind applications to higher-spin gauge theories we use forms and work in the first order formalism where duality is also manifested at the level of the action [16]. Moreover, the first order formalism is relevant for applications of duality to holography, since the correlation functions of the boundary theory are essentially determined by the bulk canonical momenta (see e.g. [17]).

Our aim in this work is to formulate linearized first order gravity using suitable "electric" and "magnetic" variables, in close analogy with electromagnetism. We find that this is possible *only* when the background geometry is Minkowski or (A)dS₄. Then we implement the standard electric-magnetic duality rotations. We find that, up to "boundary" terms, the linearized Hamiltonian changes by terms that do not alter the bulk dynamics i.e. do not alter the second order bulk equations of motion. Moreover, the duality rotation interchanges the (linearized) constraints with the (linearized) Bianchi identities. The "boundary" terms have important holographic consequences since they correspond to marginal "double-trace" deformations [3] that induce the boundary $SL(2, \mathbb{Z})$ action. In the Appendix we exhibit a modified duality rotation that leaves the bulk Hamiltonian invariant and induces "boundary" terms that correspond to relevant deformations as in [3].

2. Action and Hamiltonian

Having in mind the extension of our results to higher-spin gauge theories we start from the MacDowell-Mansouri form [18] of the gravitational action⁴

$$I_{MM} = \frac{1}{2\Lambda} \int_M \epsilon_{abcd} (R^{ab} \wedge R^{cd} + 2\Lambda e^a \wedge e^b \wedge R^{cd} + \Lambda^2 e^a \wedge e^b \wedge e^c \wedge e^d) , \quad (2.1)$$

³An interesting formulation of first order duality for linearized gravity around flat space was presented in [7].

⁴We note $I = -16\pi G_N S$, where S is the usually normalized gravitational action.

where a, b, \dots are Lorentz indices. In this formalism, the vierbein e^a and the spin connection ω^a_b are initially thought of as independent variables. The curvature 2-form is

$$R^a_b = d\omega^a_b + \omega^a_c \wedge \omega^c_b = \frac{1}{2} R^a_{bcd} e^c \wedge e^d.$$

Varying the action with respect to e^a and ω^a_b , we find

$$R^{ab} + \Lambda e^a \wedge e^b = 0, \quad (2.2)$$

$$T^a = de^a + \omega^a_b \wedge e^b = 0. \quad (2.3)$$

The relation to gravity is established via the vanishing torsion equation (2.3), which relates e and ω in the familiar way. The above equations are equivalent to the Einstein equation in metric variables

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = +3\Lambda g_{\mu\nu}. \quad (2.4)$$

and the scalar curvature is $R = -d(d-1)\Lambda = -12\Lambda$. Note that our Λ is related to the cosmological constant in its usual definition via $\Lambda_{cosm} = -6\Lambda$. $\Lambda > 0$ corresponds to AdS .

Note that this is actually $SO(3, 2)$ covariant, as we can combine ω, e into a super-connection. Note that Λ has units $(Length)^{-2}$. In the $SO(3, 2)$ -invariant formalism, I_{MM} arises from

$$I_{MM} = \frac{1}{2\Lambda} \int \epsilon_{ABCDE} V^E \mathcal{R}^{AB} \wedge \mathcal{R}^{CD}, \quad (2.5)$$

where V^E is a non-dynamical 0-form field (that we take to have value $V^{-1} = 1$ to gauge back to the $SO(3, 1)$ formalism) and \mathcal{R}^A_B is the curvature of $\Omega^A_B \equiv \{e^a, \omega^a_b\}$. There are also quasi-topological terms of the form

$$I_{top} = \frac{\theta}{2\Lambda} \int \mathcal{R}^A_B \wedge \mathcal{R}^B_A + \frac{\alpha}{\Lambda} \int \mathcal{R}^A_B \wedge \mathcal{R}_{AC} V^B V^C \quad (2.6)$$

that we could add to the action. In the stated gauge, this reduces to

$$I_{top} = \frac{\theta}{2\Lambda} P_2 + (\theta + \alpha) C_{NY} + \alpha \int R_{ab} \wedge e^a \wedge e^b \quad (2.7)$$

where $P_2 = \int R^a_b \wedge R^b_a$ is the Pontryagin class, $C_{NY} = \int (T^a \wedge T_a - R_{ab} \wedge e^a \wedge e^b)$ is the Nieh-Yan class and we also note the Euler class $E_2 = \int \epsilon_{abcd} R^{ab} \wedge R^{cd}$. Note that in the presence of torsion, the action (2.7) contains the non-topological term $\int R_{ab} \wedge e^a \wedge e^b$ with ‘‘Immirzi parameter’’ $\gamma = -2/\alpha$. In the absence of torsion, this term is a total derivative.

The Hilbert-Palatini action is

$$I_{HP} = I_{MM} - \frac{1}{2\Lambda} E_2. \quad (2.8)$$

It differs from I_{MM} by a boundary term, is smooth as $\Lambda \rightarrow 0$ but is not manifestly $SO(3, 2)$ -invariant.

2.1 The 3 + 1 split

Next, we carefully consider the 3 + 1 split. Although much of the discussion here is familiar from the ADM formalism, we feel it is important to set notation carefully, as we will introduce some new ingredients. To accommodate both *AdS* and *dS* signatures simultaneously, we will introduce a ‘time’ function t and a foliation of space-time $\Sigma_t \hookrightarrow M$. In *dS*, t is time-like, and this corresponds to the usual Hamiltonian foliation; in *AdS* on the other hand, we will take t to be the (space-like) radial coordinate. We will keep track of the resulting signs by a parameter σ_\perp , equal to ± 1 in *dS*(*AdS*).

Proceeding as usual then, we get a vector field \mathbf{t} that satisfies $\nabla_{\mathbf{t}} t = 1 \equiv \mathbf{t}(t)$ (so $\mathbf{t} = \frac{\partial}{\partial t}$) and a 1-form dt . Given a 4-metric, we can introduce the normal 1-form n as

$$n = \sigma_\perp N dt, \quad (2.9)$$

which is normalized as $(n, n) = \sigma_\perp$. The dual vector field \mathbf{n} can be expanded as

$$\mathbf{n} = \frac{1}{N} \mathbf{t} - \frac{1}{N} \mathbf{N}, \quad (2.10)$$

where the shift \mathbf{N} satisfies $(\mathbf{N}, \mathbf{n}) = 0$, and thus $(\mathbf{t}, \mathbf{n}) = \sigma_\perp N$.

Next, we will locally choose a basis of 1-forms

$$e^0 = \sigma_\perp n = N dt, \quad (2.11)$$

$$e^\alpha = \tilde{e}^\alpha + N^\alpha dt. \quad (2.12)$$

The \tilde{e}^α span $T^*\Sigma_t$, and correspond to a 3-metric $h_{ij} = \tilde{e}_i^\alpha \tilde{e}_j^\beta \eta_{\alpha\beta}$. The quantities N^α are the components of \mathbf{N} : $N^\alpha = e_i^\alpha N^i$. These basis 1-forms are dual to $\{\mathbf{e}_0 = \mathbf{n}, \mathbf{e}_\alpha = \tilde{\mathbf{e}}_\alpha\}$, with $\mathbf{e}_a(e^b) = \delta_a^b$.

We expand the spin connection in the same basis⁵

$$\omega^a{}_b = q^a{}_b dt + \tilde{\omega}^a{}_b, \quad (2.13)$$

which leads to

$$R^a{}_b = \tilde{R}^a{}_b + dt \wedge r^a{}_b, \quad (2.14)$$

where \tilde{R} is formed from $\tilde{\omega}$ and \tilde{d} only, and

$$r^a{}_b = \dot{\tilde{\omega}}^a{}_b - \tilde{d}q^a{}_b - \tilde{\omega}^a{}_c q^c{}_b + q^a{}_c \tilde{\omega}^c{}_b. \quad (2.15)$$

Note that these quantities are merely decompositions along $T^*\Sigma_t$ in the 4-geometry; we will introduce the intrinsically defined objects shortly.

We then find

$$I_{HP} = 2\epsilon_{\alpha\beta\gamma} \int dt \wedge \left\{ N(\tilde{R}^{\alpha\beta} + \Lambda \tilde{e}^\alpha \wedge \tilde{e}^\beta) \wedge \tilde{e}^\gamma - 2N^\alpha(\tilde{R}^{0\beta}) \wedge \tilde{e}^\gamma + r^{0\alpha} \wedge \tilde{e}^\beta \wedge \tilde{e}^\gamma \right\}. \quad (2.16)$$

⁵We have $q^a{}_b = N\omega_0^a{}_b + N^\alpha\omega_\alpha^a{}_b$ and $\tilde{\omega}_b^a \equiv \omega_\alpha^a{}_b \tilde{e}^\alpha$.

As is familiar, the lapse and shift appear as Lagrange multipliers. The constraints that they multiply are of course zero in any background (i.e. vacuum solution), such as (A)dS₄. The final term in the action contains the real dynamics – $r^{0\alpha}$ depends on the components $R^{0\alpha}_{0\beta}$ of the Riemann tensor.

Note though that the tensors used here are 4-dimensional. Let us define the "electric field"

$$K_\alpha = \sigma_\perp \tilde{\omega}^0{}_\alpha = K_{\beta\alpha} \tilde{e}^\beta. \quad (2.17)$$

In the case that ω is the torsion-free Levi-Civita connection, this agrees with the standard definition for extrinsic curvature, regarded as a vector-valued one-form. We then find

$$\tilde{R}^\alpha{}_\beta = {}^{(3)}R^\alpha{}_\beta - \sigma_\perp K^\alpha \wedge K_\beta, \quad (2.18)$$

and

$$\tilde{R}^0{}_\alpha = \sigma_\perp (\tilde{d}K_\alpha + K_\beta \wedge \tilde{\omega}^\beta{}_\alpha) \equiv \sigma_\perp (\tilde{D}K)_\alpha. \quad (2.19)$$

These equations amount to the Gauss-Codazzi relations.

Furthermore, $r^{0\alpha}$ contains time derivatives of $\tilde{\omega}^{0\alpha}$ as well as terms linear in components of q . We find

$$\begin{aligned} 2\epsilon_{\alpha\beta\gamma} r^{0\alpha} \wedge \tilde{e}^\beta \wedge \tilde{e}^\gamma &= 2\epsilon_{\alpha\beta\gamma} \left[\sigma_\perp \dot{K}^\alpha - (\tilde{D}q)^{0\alpha} \right] \wedge \tilde{e}^\beta \wedge \tilde{e}^\gamma, \\ &= 2\sigma_\perp \epsilon_{\alpha\beta\gamma} \left(\dot{K}^\alpha + q^{\alpha\delta} K_\delta \right) \wedge \tilde{e}^\beta \wedge \tilde{e}^\gamma + 4q^{0\alpha} \left[\epsilon_{\alpha\beta\gamma} \tilde{T}^\beta \wedge \tilde{e}^\gamma \right] \end{aligned} \quad (2.20)$$

up to a total 3-derivative. We have defined the intrinsic 3-torsion $\tilde{T}^\alpha = \tilde{d}\tilde{e}^\alpha + \tilde{\omega}^\alpha{}_\beta \wedge \tilde{e}^\beta$. Since we wish to regard the \tilde{e} as coordinate variables,⁶ we integrate the first term by parts to obtain (up to the total time-derivative $\frac{\partial}{\partial t} (2\sigma_\perp K^\alpha \wedge \tilde{e}^\beta \wedge \tilde{e}^\gamma \epsilon_{\alpha\beta\gamma})$)

$$2\epsilon_{\alpha\beta\gamma} r^{0\alpha} \wedge \tilde{e}^\beta \wedge \tilde{e}^\gamma = \Pi_\alpha \wedge \dot{\tilde{e}}^\alpha + 4q^{0\alpha} \epsilon_{\alpha\beta\gamma} \tilde{T}^\beta \wedge \tilde{e}^\gamma + 2\sigma_\perp q^{\alpha\delta} \epsilon_{\alpha\beta\gamma} K_\delta \wedge \tilde{e}^\beta \wedge \tilde{e}^\gamma. \quad (2.21)$$

where we have defined the momentum 2-form

$$\Pi_\alpha = -4\sigma_\perp \epsilon_{\alpha\beta\gamma} K^\beta \wedge \tilde{e}^\gamma. \quad (2.22)$$

The q^{ab} appear as Lagrange multipliers. In particular, the $q^{\alpha\beta}$ constraint precisely sets the antisymmetric (torsional) part of the extrinsic curvature tensor $K_{[\alpha\beta]}$ to zero. Next, we define the "magnetic field"

$$B_\alpha = \frac{1}{2} \sigma_\perp \epsilon_{\alpha\beta\gamma} \tilde{\omega}^{\beta\gamma}, \quad \omega^{\alpha\beta} = -\epsilon^{\alpha\beta\gamma} B_\gamma. \quad (2.23)$$

and we find that the $q^{0\alpha}$ constraint

$$\epsilon_{\alpha\beta\gamma} \tilde{T}^\beta \wedge \tilde{e}^\gamma = \epsilon_{\alpha\beta\gamma} \tilde{d}\tilde{e}^\beta \wedge \tilde{e}^\gamma - \sigma_\perp B_\beta \wedge \tilde{e}^\beta \wedge \tilde{e}_\alpha = 0, \quad (2.24)$$

⁶Without this integration by parts, we would be in the Ashtekar formalism. Here, our choice gives a formalism closely related to the metric variable formalism. Note that the induced boundary term may be written $-\frac{1}{2}\Pi_\alpha \wedge \tilde{e}^\alpha$.

involves only the antisymmetric part $B_{[\alpha,\beta]}$ of the magnetic field $B_\alpha = B_{\alpha\beta}\tilde{e}^\beta$. The antisymmetric part of B_α spoils the gauge covariance of the constraint (2.24) under an $\text{SO}(3)$ rotation of the dreibein \tilde{e}^α , hence it represents degrees of freedom that can be gauged fixed to zero by an $\text{SO}(3)$ rotation. On the other hand, an algebraic equation of motion connects the symmetric part of $B_{\alpha\beta}$ to derivatives of \tilde{e}^α as

$$\tilde{d}\tilde{e}^\alpha + \epsilon^{\alpha\beta\gamma} B_\beta \wedge \tilde{e}_\gamma = 0 \quad (2.25)$$

At the end, one is left with the canonically conjugate variables \tilde{e}^α and Π_α . These results are familiar from the metric formalism.

Dropping the torsional terms, we then arrive at the action

$$I_{HP} = \int dt \wedge \left\{ \dot{\tilde{e}}^\alpha \wedge \Pi_\alpha + 2N\epsilon_{\alpha\beta\gamma}({}^{(3)}R^{\alpha\beta} - \sigma_\perp K^\alpha \wedge K^\beta + \Lambda\tilde{e}^\alpha \wedge \tilde{e}^\beta) \wedge \tilde{e}^\gamma - 4\sigma_\perp N^\alpha \epsilon_{\alpha\beta\gamma}(\tilde{D}K)^\beta \wedge \tilde{e}^\gamma \right\}. \quad (2.26)$$

Furthermore, using $*_3\tilde{e}^\alpha \wedge \tilde{e}^\beta = \frac{1}{2}\eta_{\gamma\delta}\epsilon^{\alpha\beta\gamma}\tilde{e}^\delta$, we have

$$\hat{\Pi}_\alpha = *_3\Pi_\alpha = -2(K_{\alpha\beta} - \eta_{\alpha\beta}trK)\tilde{e}^\beta, \quad (2.27)$$

where $trK = \eta^{\alpha\beta}K_{\alpha\beta}$. We can solve the above equation to get

$$K_\alpha = -\frac{1}{2}(\hat{\Pi}_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}tr\hat{\Pi})\tilde{e}^\beta. \quad (2.28)$$

As stated above, $K_{\alpha\beta}$ (and $\hat{\Pi}_{\alpha\beta}$) is symmetric when the torsion vanishes.

Finally, with the definition (2.23) we find⁷

$$\epsilon_{\alpha\beta\gamma}({}^{(3)}R^{\alpha\beta} \wedge \tilde{e}^\gamma = \epsilon_{\alpha\beta\gamma} \left[\tilde{d}\tilde{\omega}^{\alpha\beta} + \tilde{\omega}^\alpha_\delta \wedge \tilde{\omega}^{\delta\beta} \right] \wedge \tilde{e}^\gamma = \sigma_\perp \left[2\tilde{d}B_\gamma + \epsilon_{\alpha\beta\gamma}B^\alpha \wedge B^\beta \right] \wedge \tilde{e}^\gamma. \quad (2.29)$$

Introducing B_α is an unusual thing to do but it will play a role in duality: in this form, the Hamiltonian contains terms which are reminiscent of those of the Maxwell theory. The full HP action is of the form

$$I_{HP} = \int dt \wedge \left\{ \dot{\tilde{e}}^\alpha \wedge \Pi_\alpha - 4\sigma_\perp N^\alpha \epsilon_{\alpha\beta\gamma}(\tilde{D}K)^\beta \wedge \tilde{e}^\gamma + 2\sigma_\perp N(2\tilde{d}B_\gamma + \epsilon_{\alpha\beta\gamma}B^\alpha \wedge B^\beta - \epsilon_{\alpha\beta\gamma}K^\alpha \wedge K^\beta + \sigma_\perp \Lambda \epsilon_{\alpha\beta\gamma}\tilde{e}^\alpha \wedge \tilde{e}^\beta) \wedge \tilde{e}^\gamma \right\}. \quad (2.30)$$

Note that the entire contribution of the cosmological constant appears in the last term of the Hamiltonian constraint.

⁷The spatial signature σ_3 appears in $\epsilon_{\alpha\beta\gamma}\epsilon_{\phi\delta\rho}\eta^{\alpha\phi} = \sigma_3(\eta_{\beta\delta}\eta_{\gamma\rho} - \eta_{\beta\rho}\eta_{\gamma\delta})$. We will always consider Lorentzian spacetime signature, so $\sigma_3 = -\sigma_\perp$.

2.2 Shifted Variables

It is possible to make a transformation of the canonical variables in order to absorb the cosmological constant term in (2.30). This can be achieved by introducing the new variables

$$\hat{K}^\alpha = K^\alpha - \rho \tilde{e}^\alpha, \quad (2.31)$$

and requiring that

$$\rho^2 = \sigma_\perp \Lambda. \quad (2.32)$$

This is positive only when σ_\perp and Λ are simultaneously positive or negative, as it is the case for both AdS_4 ($\Lambda > 0$) and dS_4 ($\Lambda < 0$). We will often write $\Lambda = \sigma_\perp / L^2$ where L is a length scale.

Under (2.31) the momentum 2-form becomes

$$\Pi_\alpha \rightarrow \mathcal{P}_\alpha - 4\sigma_\perp \rho \epsilon_{\alpha\beta\gamma} \tilde{e}^\beta \wedge \tilde{e}^\gamma. \quad (2.33)$$

The last term in (2.33) contributes a total time derivative to the action (of the form of a boundary cosmological term). We have introduced a new momentum variable

$$\mathcal{P}_\alpha = -4\sigma_\perp \epsilon_{\alpha\beta\gamma} \hat{K}^\beta \wedge \tilde{e}^\gamma.$$

Then, we get the action

$$\begin{aligned} I_{HP} = \int dt \wedge \left\{ \dot{\tilde{e}}^\alpha \wedge \mathcal{P}_\alpha - 4\sigma_\perp N^\alpha \epsilon_{\alpha\beta\gamma} (\tilde{D}\hat{K} + \rho \tilde{T})^\beta \wedge \tilde{e}^\gamma - \frac{4}{3} \sigma_\perp \rho \epsilon_{\alpha\beta\gamma} \frac{\partial}{\partial t} (\tilde{e}^\alpha \wedge \tilde{e}^\beta \wedge \tilde{e}^\gamma) \right. \\ \left. + 2\sigma_\perp N \left(2\tilde{d}(B_\alpha \wedge \tilde{e}^\alpha) + 2B_\gamma \wedge \tilde{T}^\gamma - \epsilon_{\alpha\beta\gamma} \left(B^\alpha \wedge B^\beta + \hat{K}^\alpha \wedge \hat{K}^\beta + 2\rho \hat{K}^\alpha \wedge \tilde{e}^\beta \right) \wedge \tilde{e}^\gamma \right) \right\}. \end{aligned} \quad (2.34)$$

Note that the shift constraint is still written in terms of the ordinary covariant derivative, and thus involves a non-linear term coupling B to \hat{K} . Consistent with our previous discussion, we drop the terms involving the torsion \tilde{T} , and disregard the boundary term to obtain

$$\begin{aligned} I_{HP} = \int dt \wedge \left\{ \dot{\tilde{e}}^\alpha \wedge \mathcal{P}_\alpha - 4\sigma_\perp N^\alpha \epsilon_{\alpha\beta\gamma} (\tilde{D}\hat{K})^\beta \wedge \tilde{e}^\gamma \right. \\ \left. + 2\sigma_\perp N \left(2\tilde{d}(B_\alpha \wedge \tilde{e}^\alpha) - \epsilon_{\alpha\beta\gamma} \left(B^\alpha \wedge B^\beta + \hat{K}^\alpha \wedge \hat{K}^\beta + 2\rho \hat{K}^\alpha \wedge \tilde{e}^\beta \right) \wedge \tilde{e}^\gamma \right) \right\}. \end{aligned} \quad (2.35)$$

We note that the parameter ρ can be of *either* sign (although, this sign does not appear in the second order equations of motion).

2.3 Linearization

Next, we linearize the above action around an appropriate fixed background. We expand as

$$\tilde{e}^\alpha = \underline{\tilde{e}}^\alpha + E^\alpha, \quad N = 1 + n, \quad N^\alpha = n^\alpha, \quad B^\alpha = \underline{B}^\alpha + b^\alpha, \quad \hat{K}^\alpha = \underline{\hat{K}}^\alpha + k^\alpha. \quad (2.36)$$

The background values should satisfy the constraints. The simplest choice is the background where

$$\underline{\hat{K}}^\alpha = 0 = \underline{B}^\alpha. \quad (2.37)$$

In fact, reaching this simple form was a motivation for the shift (2.31). Then, to quadratic order in the fluctuating fields the Hamiltonian gives

$$I_{HP} = \int dt \wedge \left\{ \dot{E}^\alpha \wedge p_\alpha - 4\sigma_\perp n^\alpha \epsilon_{\alpha\beta\gamma} \tilde{d}k^\beta \wedge \tilde{e}^\gamma + 4\sigma_\perp n \left(\tilde{d}(b_\alpha \wedge \tilde{e}^\alpha) - \rho \epsilon_{\alpha\beta\gamma} k^\alpha \wedge \tilde{e}^\beta \wedge \tilde{e}^\gamma \right) \right. \\ \left. - 2\sigma_\perp \epsilon_{\alpha\beta\gamma} (b^\alpha \wedge b^\beta + k^\alpha \wedge k^\beta + 2\rho k^\alpha \wedge E^\beta) \wedge \tilde{e}^\gamma \right\} \quad (2.38)$$

where

$$p_\alpha = -4\sigma_\perp \epsilon_{\alpha\beta\gamma} k^\beta \wedge \tilde{e}^\gamma \quad (2.39)$$

are the linearized momentum variables conjugate to E^α .

In order to reach the form (2.38) the linear terms in the fluctuations must vanish. For this to happen we find the relationships

$$\dot{\tilde{e}}^\alpha + \rho \tilde{e}^\alpha = 0. \quad (2.40)$$

Notice that we can also write the linearized action in the form

$$I_{HP} = \int dt \wedge \left\{ (\dot{E}^\alpha + \rho E^\alpha) \wedge p_\alpha - 2\sigma_\perp \epsilon_{\alpha\beta\gamma} (b^\alpha \wedge b^\beta + k^\alpha \wedge k^\beta) \wedge \tilde{e}^\gamma \right. \\ \left. - 4\sigma_\perp n^\alpha \epsilon_{\alpha\beta\gamma} \tilde{d}k^\beta \wedge \tilde{e}^\gamma + n \left(4\sigma_\perp \tilde{d}b_\gamma + \rho p_\gamma \right) \wedge \tilde{e}^\gamma \right\}. \quad (2.41)$$

The form of the first term, involving the momentum, makes clear that longitudinal fluctuations are non-dynamical. The natural time dependence of E^α is of the form $e^{-\rho t}$ (correspondingly, the natural time dependence of p_α is $e^{+\rho t}$). Other than that, we see that in comparing to the flat space action, in these variables, the only change is that the Hamiltonian constraint is modified.

The solutions of (2.40) and (2.37) are components of (A)dS₄ spacetimes. We can solve (2.40) to obtain

$$\underline{e}^0 = dt, \quad \underline{e}^\alpha = e^{-\rho t} dx^\alpha. \quad (2.42)$$

With these we construct the usual Poincaré metric on (A)dS which, however, covers only half of the space even though the parameter t runs from $-\infty$ to $+\infty$. The conformal boundary in these coordinates is at $t = +\infty$. Then we derive

$$\underline{\omega}^\alpha{}_0 = -\rho e^{-\rho t} dx^\alpha = -\rho \underline{e}^\alpha, \quad (2.43)$$

and so

$$\left. \begin{aligned} \underline{R}^\alpha{}_\beta &= -\frac{\sigma_\perp}{L^2} \underline{e}^\alpha \wedge \underline{e}_\beta \\ \underline{R}^\alpha{}_0 &= -\frac{1}{L^2} \underline{e}^\alpha \wedge \underline{e}^0 \end{aligned} \right\} \underline{R}^a{}_b = -\frac{\sigma_\perp}{L^2} \underline{e}^a \wedge \underline{e}_b. \quad (2.44)$$

Hence $\underline{Ric}_{ab} = -\frac{3\sigma_\perp}{L^2} \eta_{ab}$ and $\underline{R} = -12\sigma_\perp/L^2 = -12\Lambda$. We also evaluate

$$\underline{\Pi}_\alpha = -4\sigma_\perp \rho \epsilon_{\alpha\beta\gamma} \tilde{e}^\beta \wedge \tilde{e}^\gamma, \quad \hat{\Pi}^\alpha = 4\rho \tilde{e}^\alpha, \quad tr \hat{\Pi} = 12\rho \quad (2.45)$$

$$\underline{B}_\alpha = 0, \quad \underline{K}^\alpha = \rho \tilde{e}^\alpha \Rightarrow \hat{K}^\alpha = 0 \quad (2.46)$$

Note that in this gauge, $(\tilde{D}\underline{K})^\alpha = \frac{1}{L} \tilde{T}^\alpha = 0$, which solves the shift constraint, while the Hamiltonian constraint is satisfied through a cancellation between the \underline{K}^2 term and the cosmological term.

3. Linearized Gravitational Duality and Holography

Let us summarize what we have obtained so far. In the presence of a cosmological constant we have defined variables such that the action resembles most closely the action without the cosmological constant. This was done in order to look for a suitable background around which linear fluctuations are as simple as possible. Requiring that \hat{K} (the “electric field”) and B (the “magnetic field”) vanish in such a background - as they do around flat space - we found that the background should be (A)dS₄. Quite satisfactorily, *both* sign choices for ρ in the change of variables (2.31) lead to (A)dS₄ spacetimes.

3.1 Duality and Holography

This is the appropriate point to recall some salient features of duality rotations. In simple Hamiltonian systems the effect of the canonical transformation $p \mapsto q$ and $q \mapsto -p$ to the action is (see e.g. [19])

$$I = \int_{t_1}^{t_2} dt [p\dot{q} - H(p, q)] \mapsto I_D = \int_{t_1}^{t_2} dt [-q\dot{p} - H(q, -p)]. \quad (3.1)$$

Notice that I_D involves the *dual* variables, for which we have however kept the same notation for simplicity. The transformed Hamiltonian $H(q, -p)$ is in general not related to $H(p, q)$. However, if $H(q, -p) = H(p, q)$ we call the above transformation a *duality*. It then holds

$$I_D = I - qp \Big|_{t_1}^{t_2}. \quad (3.2)$$

The dual action describes exactly the same dynamics as the initial one, up to a modification of the boundary conditions. For example, if I is stationary on the e.o.m for fixed q in the boundary, I_D is stationary on the same e.o.m. for fixed p in the boundary. This simple example illustrates the role of duality in holography; a bulk duality transformation corresponds to a particular modification of the boundary conditions. This property of duality transformations is behind the remarkable holographic properties of electromagnetism in (A)dS₄ [2, 3].

Clearly, the crucial properties of a duality transformation are to be canonical and to leave the Hamiltonian unchanged. However, consider a slight generalization

$$S = \int_{t_1}^{t_2} dt [p\dot{q} - \frac{1}{2}(p^2 + q^2 + 2\lambda pq)] \quad (3.3)$$

where λ is an arbitrary parameter. The Hamiltonian now is *not* invariant under the canonical transformation $p \mapsto q$ and $q \mapsto -p$ - the pq term changes sign. Consequently, the first order form of the equations of motion are also not duality invariant. Nevertheless, the second order equation of motion is invariant. We will find that gravity in the presence of a cosmological constant follows precisely this model. Of course, gravity is a much more complicated constrained system, but as we will show, the constraints and Bianchi identities transform appropriately.

We also note that the canonical transformation (implemented by a generating functional of the first kind)

$$p \mapsto q + 2\lambda p, \quad q \mapsto -p. \quad (3.4)$$

is of interest here. The above does not change the Hamiltonian and the transformed action differs from the initial one by total time derivative terms⁸

$$S \mapsto S_D = S - pq \Big|_{t_1}^{t_2} - \lambda p^2 \Big|_{t_1}^{t_2}. \quad (3.5)$$

3.2 Linearized gravitational duality

As a preamble to gravity we recall the duality properties of Maxwell theory

$$I_{Max} = \frac{1}{2g^2} \int dt \wedge \left\{ \dot{\tilde{A}} \wedge *_3 E - \frac{1}{2} (E \wedge *_3 E + B \wedge *_3 B) - A_0 \tilde{d} *_3 E \right\}, \quad (3.6)$$

Under the duality $E \mapsto - *_3 B$, $B \mapsto *_3 E$, $\tilde{A} \mapsto \tilde{A}_D$, we find

$$I_{Max} \mapsto I_{Max,D} = \frac{1}{2g^2} \int dt \wedge \left\{ -\dot{\tilde{A}}_D \wedge B - \frac{1}{2} (E \wedge *_3 E + B \wedge *_3 B) + A_0 \tilde{d} B \right\}. \quad (3.7)$$

E and B in (3.7) should be expressed through \tilde{A}_D . We observe that the kinetic term has changed sign, while the Hamiltonian remains invariant. In addition, the (Gauss) constraint is dualized to the trivial ‘Bianchi’ identity $dB = 0$ for the dual magnetic field.

Next we try to apply a Maxwell-type duality map in gravity. We consider the following transformation around the fixed background (2.40)

$$k^\alpha \mapsto -b^\alpha, \quad b^\alpha \mapsto k^\alpha. \quad (3.8)$$

To implement the map (3.8) we need to specify the mapping of E^α to a ‘dual 3-bein’ \mathcal{E}^α . We do that using the linearized form of (2.25) as

$$\epsilon^{\alpha\beta\gamma} b_\beta \wedge \tilde{\underline{e}}_\gamma + \tilde{d}E^\alpha = 0 \mapsto \epsilon^{\alpha\beta\gamma} k_\beta \wedge \tilde{\underline{e}}_\gamma + \tilde{d}\mathcal{E}^\alpha = 0 = \tilde{d}\mathcal{E}^\alpha - \frac{1}{4\sigma_\perp} p_\alpha \quad (3.9)$$

Since $p_\alpha = 4\sigma_\perp \tilde{d}\mathcal{E}_\alpha$, it is natural to define

$$p_{D,\alpha} = 4\sigma_\perp \tilde{d}E_\alpha = -4\sigma_\perp \epsilon_{\alpha\beta\gamma} b^\beta \wedge \tilde{\underline{e}}^\gamma, \quad (3.10)$$

and thus the mapping (3.8) is supplemented by

$$E \mapsto \mathcal{E}, \quad \mathcal{E} \mapsto -E, \quad p \mapsto -p_D, \quad p_D \mapsto p \quad (3.11)$$

⁸In holography, the latter terms correspond to the relevant “multi-trace” boundary deformations discussed in [3].

Now, let us see the effects of the above duality mapping. The action transforms to

$$I_{HP} \mapsto I_{HP,D} = \int dt \wedge \left\{ -\dot{\mathcal{E}}^\alpha \wedge p_{D,\alpha} - \rho \mathcal{E}^\alpha \wedge p_{D,\alpha} - 2\sigma_\perp \epsilon_{\alpha\beta\gamma} (b^\alpha \wedge b^\beta + k^\alpha \wedge k^\beta) \wedge \tilde{\underline{e}}^\gamma \right. \\ \left. + 4\sigma_\perp n^\alpha \epsilon_{\alpha\beta\gamma} \tilde{d}b^\beta \wedge \tilde{\underline{e}}^\gamma + n \left(4\sigma_\perp \tilde{d}k_\gamma + \rho p_{D,\alpha} \right) \wedge \tilde{\underline{e}}^\gamma \right\} \quad (3.12)$$

where now k^α and b^α should be expressed in terms of the dual variables \mathcal{E}^α and $p_{D,\alpha}$ via (3.9) and (3.10). We notice that the 'kinetic' part $\dot{\mathcal{E}} \wedge p$ of the action changes sign under the duality map, in direct analogy with the Maxwell case. However, the Hamiltonian is not invariant due to the change of sign of the second term in the first line of (3.12). We will discuss this further in a later section. For now, we note that this sign change would not show up in the equations of motion, *written in second order form*. It is important to also note that the constraints are transformed into quantities which in the next subsection we will recognize as the linearized Bianchi identities. This is to be expected since the duality transformations are canonical. We also note that it may be possible to choose an alternative canonical transformation, designed to leave the Hamiltonian invariant. The latter is presumably related to the work of Julia et. al. [15] and is considered in the Appendix.

3.3 Linearized Constraints and Bianchi Identities

By virtue of the discussion above we may now demonstrate that under the duality mapping (3.8) the linearized constraints transform to the linearized Bianchi identities as

$$C_\alpha \equiv \epsilon_{\alpha\beta\gamma} \tilde{d}k^\beta \wedge \tilde{\underline{e}}^\gamma \mapsto -\epsilon_{\alpha\beta\gamma} \tilde{d}b^\beta \wedge \tilde{\underline{e}}^\gamma \quad (3.13)$$

$$C_0 \equiv -\sigma_\perp \left(\tilde{d}b_\gamma - \rho \epsilon_{\alpha\beta\gamma} k^\alpha \wedge \tilde{\underline{e}}^\beta \right) \wedge \tilde{\underline{e}}^\gamma \mapsto -\sigma_\perp \left(\tilde{d}k_\gamma + \rho \epsilon_{\alpha\beta\gamma} b^\alpha \wedge \tilde{\underline{e}}^\beta \right) \wedge \tilde{\underline{e}}^\gamma \quad (3.14)$$

To identify the right hand sides, we first note that the Bianchi identities are

$$B_R^a{}_b = dR^a{}_b - R^a{}_c \wedge \omega^c{}_b + \omega^a{}_c \wedge R^c{}_b = 0 \quad (3.15)$$

$$B_T^a = dT^a - R^a{}_b \wedge e^b + \omega^a{}_b \wedge T^b = 0 \quad (3.16)$$

which are obtained from the definitions of $R^a{}_b$ and T^a by exterior differentiation. The first equation is satisfied identically. Since the torsion vanishes, the second equation tells us only that $R^a{}_b \wedge e^b = 0$. If we do the 3+1 split, we find two equations. The first is

$$B_T^\alpha = -({}^{(3)}R^\alpha{}_\beta - \sigma_\perp K^\alpha \wedge K_\beta) \wedge \tilde{e}^\beta = 0 \quad (3.17)$$

which upon using the symmetry of K^α linearizes to

$$B_T^\alpha = -\epsilon_{\alpha\beta\gamma} \tilde{d}b^\beta \wedge \tilde{\underline{e}}^\gamma + \dots \quad (3.18)$$

Note that this is the image under duality of the shift constraint as in (3.13).

The second identity is

$$B_T^0 = -\tilde{R}^0{}_\alpha \wedge \tilde{e}^\alpha = -\sigma_\perp (\tilde{D}K)_\alpha \\ = -\sigma_\perp \left(\tilde{d}k_\alpha + \rho \epsilon_{\alpha\beta\gamma} b^\beta \wedge \tilde{\underline{e}}^\gamma \right) \wedge \tilde{\underline{e}}^\alpha = 0 \quad (3.19)$$

where to arrive in the second line we used (2.46). This is the image of the Hamiltonian constraint as in (3.14).

Summarizing, the duality transformations between linearized constraints and Bianchi identities are

$$C_\alpha \mapsto B_{T,\alpha} \quad C_0 \mapsto B_T^0 \quad (3.20)$$

$$B_{T,\alpha} \mapsto -C_\alpha \quad B_T^0 \mapsto -C_0 \quad (3.21)$$

3.4 Connection with other known dualities

The Maxwell-type duality operation (3.8) is closely related to the dualization of the first two indices of the Riemann tensor as⁹

$$R^a{}_b \rightarrow S^a{}_b \equiv \frac{1}{2} \epsilon^a{}_{bc}{}^d R^c{}_d \quad (3.22)$$

at least at the linearized level. Let us investigate (3.22) by rewriting expressions in the 3+1 split. We have

$$R^a{}_b = \tilde{R}^a{}_b + dt \wedge r^a{}_b$$

$$S^a{}_b = \tilde{S}^a{}_b + dt \wedge s^a{}_b$$

We begin with the spatial 2-forms when we have

$$\tilde{R}^{\alpha\beta} = -\epsilon^{\alpha\beta\gamma} \tilde{d}B_\gamma + \sigma_\perp (B^\alpha \wedge B^\beta - K^\alpha \wedge K^\beta) \quad (3.23)$$

$$\tilde{R}^0{}_\alpha = \sigma_\perp (\tilde{d}K_\alpha + K_\beta \wedge \tilde{\omega}^\beta{}_\alpha) \equiv \sigma_\perp (\tilde{D}K)_\alpha \quad (3.24)$$

and

$$\tilde{S}^0{}_\gamma = \frac{1}{2} \sigma_\perp \epsilon_{\alpha\beta\gamma} \tilde{R}^{\alpha\beta} \quad (3.25)$$

$$\tilde{S}_{\alpha\beta} = \epsilon_{\alpha\beta\gamma} \tilde{R}^{0\gamma} \quad (3.26)$$

If we linearize these expressions, we find under the duality transformation (3.8)

$$\tilde{R}^{ab} \mapsto -\sigma_\perp \tilde{S}^{ab} \quad (3.27)$$

Because the expressions (3.24) involve derivatives of B and K , the duality (3.8) is an ‘integrated form’ of the usual Riemann tensor duality, but implies it.

Similarly, if we investigate the spatial 1-forms, we find

$$r^{ab} \mapsto -\sigma_\perp s^{ab} \quad (3.28)$$

To arrive at this result we have set to zero the Lagrange multiplier field q .

⁹For a discussion of the duality properties of gravity in terms of the Riemann tensor see [10].

4. The Effect on the Boundary Theory

It is well known that AdS is holographic. We may well ask, in the context of AdS/CFT, how the duality transformation that we have defined here acts in the boundary. We are instructed to consider the on-shell bulk action as a function of bulk fields. So, we evaluate the action on a solution to the equation of motion, resulting in a pure boundary term which is of the form

$$S_{bdy} = \int_{\partial M} p_\alpha \wedge E^\alpha \quad (4.1)$$

Applying the duality transformation to the bulk theory, although the bulk action is not invariant as we have discussed above, nevertheless it may be easily shown that it induces a simple transformation on the (linearized) boundary term: it simply changes its sign.

$$S_{bdy}^{dual} = - \int_{\partial M} p_{D,\alpha} \wedge \mathcal{E}^\alpha \quad (4.2)$$

This transformation is exactly analogous to what happens in the Maxwell case: it amounts to the result [21].¹⁰

$$G_2 G_2^{dual} = -1. \quad (4.3)$$

5. Conclusions and Outlook

Motivated by possible application in holography and in higher-spin gauge theory we have studied the duality properties of gravity in the Hamiltonian formulation. We have presented the gravity action in terms of suitable variables that closely resemble the electric and magnetic fields in Maxwell theory. We have found suitable "electric" and "magnetic" field variables, such that at the linearized level first order gravity most closely resembles electromagnetism. This can be done *only* around Minkowski and (A)dS₄ backgrounds.

We have implemented duality transformations in the linearized gravity fluctuations around these backgrounds. In the presence of a cosmological constant, the Hamiltonian changes, nevertheless the bulk dynamics remains unaltered, while the linearized lapse and shift constraints are mapped into the linearized Bianchi identities. Moreover, the duality transformations induce boundary terms whose relevance in holography we have briefly discussed. Finally, we have exhibited a modified duality rotation that leaves the bulk Hamiltonian invariant, while it induces boundary terms corresponding to relevant deformations.

The main implication of our results is that certain properties of correlations functions in three-dimensional CFTs mimic the duality of gravity. It would be interesting to extend our results to black-hole backgrounds and also when topological terms are present in the bulk. We also expect that one can analyze the duality of higher-spin gauge theories based on our first-order approach.

Acknowledgments

¹⁰See also [22] for an interesting recent application of this formula.

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6. Appendix: other duality mappings

It is possible to find a transformation that leaves the Hamiltonian unchanged. Consider the following transformation in the fixed background (2.40)

$$k^\alpha \mapsto -b^\alpha - 2\rho\mathcal{E}^\alpha, \quad b^\alpha \mapsto k^\alpha. \quad (6.1)$$

The mapping to the dual dreibein is still specified by (3.9). A straightforward calculation reveals that the action transforms as

$$\begin{aligned} I_{HP} \mapsto I_{HP,D} = I_{HP} &+ 4\sigma_\perp \int_{\partial M} (\epsilon_{\alpha\beta\gamma}\mathcal{E}^\alpha \wedge b^\beta + \rho\epsilon_{\alpha\beta\gamma}\mathcal{E}^\alpha \wedge \mathcal{E}^\beta) \wedge \tilde{\underline{e}}^\gamma \\ &+ \int dt \wedge \left[4\sigma_\perp n^\alpha \epsilon_{\alpha\beta\gamma} \tilde{d}b^\beta \wedge \tilde{\underline{e}}^\gamma - 8\rho n^\alpha k_\beta \wedge \tilde{\underline{e}}_\alpha \wedge \tilde{\underline{e}}^\beta \right. \\ &\left. + n(4\sigma_\perp \tilde{d}k_\alpha + 4\sigma_\perp \rho\epsilon_{\alpha\beta\gamma} b^\beta \wedge \tilde{\underline{e}}^\gamma + 8\Lambda\epsilon_{\alpha\beta\gamma}\mathcal{E}^\beta \wedge \tilde{\underline{e}}^\gamma) \wedge \tilde{\underline{e}}^\alpha \right] \end{aligned} \quad (6.2)$$

The transformations (6.1) leaves unchanged the Hamiltonian and changes the action by the total "time" derivative terms shown in the first line of (6.2).

Moreover, the linearized constraints transform into the linearized Bianchi identities. Let us see that in some detail. The second term in the shift constraint is zero since k_α is a symmetric one form $k_\alpha = k_{\alpha\beta}\tilde{\underline{e}}^\beta$ with $k_{\alpha\beta} = k_{\beta\alpha}$; see (2.21).

The term proportional to Λ in the lapse constraint is also zero. This is slightly more involved to see and it is based on the possibility of solving (3.9) for \mathcal{E}^α after gauge fixing.¹¹ One way to see this is in components. Write $\mathcal{E}^\alpha = \mathcal{E}^\alpha_\beta \tilde{\underline{e}}^\beta$ and (3.9) becomes

$$\partial_\alpha \mathcal{E}^\beta_\gamma - \partial_\gamma \mathcal{E}^\beta_\alpha = \epsilon^\beta_{\delta\alpha} k^\delta_\gamma - \epsilon^\beta_{\delta\gamma} k^\delta_\alpha \quad (6.3)$$

In the "Lorentz gauge" where $\partial^\alpha \mathcal{E}^\beta_\alpha = 0 = \partial^\alpha k^\beta_\alpha$ the above can be inverted as

$$\mathcal{E}^\alpha_\beta = \frac{1}{\partial^2} \epsilon^\alpha_{\delta\gamma} \partial^\gamma k^\delta_\beta \quad (6.4)$$

Using (6.4) one verifies that the last term in the lapse constraint vanishes. This modified duality transformation is probably related to the one considered by Julia et. al. in [15].

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¹¹This is the equivalent of inverting $\bar{E} = \nabla \times \bar{A}$ in the discussion of duality in electromagnetism [16].

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